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STRATIFIED JET FLOW

by Prem K. Khosla

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Preamble

The present report is a preliminary attempt to distinguish various flow regimes in a jet flow of a stratified fluid. It points out the difficulties associated with the linear stratification model in an unbounded fluid. It seems that the modeling mechanism of generating linear stratification is quite important, and that the flow structure can change according to whether bounding walls are used to generate the stratification thermally or not. Careful evaluation of this aspect of the interaction both experimentally and analytically is needed to clarify the meaning of linear stratification in an unbounded fluid.

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INTRODUCTION

There has been considerable interest in the flow of non-homogeneous fluids. Various aspects of such flows are well documented in the book by Yih.

Martin and Long and Pao have studied the laminar flow past a flat plate. For buoyancy and viscous dominated flows these authors showed the presence of upstream wakes. Kelley and Redkopp have examined the high Reynolds number flow past the flat plate by using the systematic expansions of the governing equations. These authors showed that for wide range of values of Russel No. (parameter representing the value of stratification) the effect of buoyancy on the flow field appears through second and higher approximations.

In the present report the laminar jet problem for a stably stratified fluid is examined. For homogeneous incompressible fluids this problem has been solved by Schlichting, Rubin and Falco examined the higher approximations by applying matched asymptotic approximations. The later authors also showed that the solution due to Schlichting is non-unique and this non-uniqueness is inherent in the problems where no characteristic lenghts are available.

BASIC EQUATIONS

It is assumed that the fluid is stably stratified and the stratification is given by linear distribution of temperature with height as:

$$T = T_0 (1 + \beta_0 y) \qquad (1)$$

The governing equations for the flow are the Navier-Stokes equations with Boussinesq approximation and may be written as:

$$\nabla \cdot \vec{q} = 0$$

$$(\vec{q} \cdot \nabla)\vec{q} = -\frac{1}{\rho_0}\nabla p + \nu_0 \nabla^2 \vec{q} - \frac{\rho}{\rho_0} g \hat{k}$$

$$(q \cdot \nabla) T = k_0 \nabla^2 T \qquad (2)$$

and $$\rho = \rho_0 \left[1 - \alpha_0 (T - T_0)\right]$$

In these equations $\vec{q}$, p, ρ and T are the velocity vector, pressure, density and temperature respectively. ν_o and k_o are the coefficients of kinematic viscosity and thermal diffusivity; α_o is the coefficient of thermal expansion ; g is the acceleration due to gravity and $\hat{k}$ is a unit vector along the vertical direction. The quantities with subscript 'o' refer to the undisturbed values at y = o. The presence of stratification makes the problem asymmetric. Also the assumption of linear stratification of temperature is justified only if $(\alpha_o \beta_o \tau_o)$ is less than unity. $(\alpha_o \beta_o \tau_o)^{-1}$ thus defines the vertical extent of the region in which linear stratification may be used. In the absence of such a constraint equation (1) leads to an inconsitency for $y \to \pm \infty$. This observation regarding the linear stratification seems to be ignored by Redkopp. Full implications of this assumption in the present context will be brought out in the later section. Various quantities are non-dimensionalized as:

$$T^* = \frac{T - T_0}{T_0} - \beta_0 L \bar{y}$$
$$\rho^* = \rho/\rho_0 = 1 - \alpha T_0 [\beta_0 L \bar{y} + T^*] \qquad (2\text{-}a)$$
$$p^* = \frac{p - p_0}{\rho_0 u_0^2} + \frac{1}{F_L}\left[\bar{y} - \tfrac{1}{2}\alpha_0 \beta_0 T_0 L \bar{y}^2\right]$$
$$\frac{u}{u_0} = \bar{u} \quad , \quad v/u_0 = \bar{v}$$
$$\frac{x}{L} = \bar{x} \quad , \quad y/L = \bar{y}$$

Now, introducing the stream function, equations (2) can be written as:

$$\left.\begin{aligned} \left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{R_e}\nabla^2\right)\nabla^2\psi &= -\frac{\alpha}{F_L} T^*_x \\ \left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{P_r R_e}\nabla^2\right) T^* &= \beta \psi_x \end{aligned}\right] \qquad (3)$$

where $\alpha = \alpha_o \tau_o$, $\beta = \beta_o L$, $F_L = U_o^2/g$ is the Froud number ; P_r is the Prandtl number and R_e the Reynolds number. Since the asymmetric part of the solution has been subtracted out while defining various non-dimensional variables, the boundary conditions for the problem can be written as:

$$\begin{aligned} \psi(x,0) = \psi_{yy}(x,0) &= 0 \quad , \quad T^*(x,0) = 0 \\ \psi_y(x,\infty) &= 0 \quad , \quad T^*(x,\infty) = 0 \end{aligned} \qquad (4)$$

In order to evaluate the effect of stratification for large Reynolds number flows, we will make use of the higher order boundary-layer theory.

INNER SOLUTION

For infinite Reynolds number the solution of equations (3) satisfying the boundary conditions at infinity is given by

$$T^* = 0 \quad , \quad \psi = 0 \qquad (5)$$

This solution does not satisfy the boundary conditions on the axis of the jet. Redefining the inner variables as

$$z = y/\epsilon(R_e) \quad ; \quad \psi = \delta(R_e)\,\Psi \quad , \text{ and}$$

substituting in equations (3), we get

$$\left[\Psi_z \frac{\partial}{\partial x} - \Psi_x \frac{\partial}{\partial z} - \frac{1}{\epsilon\delta R_e}\left(\epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right]\left(\epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Psi = -\left(\frac{\epsilon^3}{\delta^2} R_e^n\right)\frac{\alpha}{\beta}\, T^*_x \qquad (6)$$

$$\left[\Psi_z \frac{\partial}{\partial x} - \Psi_x \frac{\partial}{\partial z} - \frac{1}{\epsilon\delta R_e P_r}\left(\epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right] T^* = \beta\epsilon\, \Psi_x$$

where $\beta / F_L = R_{UL} = R_e^n$ (Russel no.)

For the jet, the constancy of the momentum flux in the x-direction is obtained from the momentum integral and is given as:

$$\frac{d}{dx}\int_{-\infty}^{\infty} \left(P^* + U^2/2\right) dy = 0 \qquad (7)$$

or

$$\int_{-\infty}^{\infty} \left(P^* + U^2/2\right) = \text{const.}$$

P^* is included in the above expression as for a buoyancy-viscous dominated jet, P^* is not small. However for inertia dominated jets, the buoyancy terms in the expression for P^* are of the same order as in equation (6). We will see that these terms are small and may be neglected in equation (6) a priori. Thus, we get

$$\frac{\delta^2}{\epsilon} \int_{-\infty}^{\infty} (\Psi_z)^2 dz = \text{const.}$$

which requires $\delta^2 = \epsilon$. Now from equations (6), one can see that for convective terms to be of the same order as viscous terms $\delta = R_e^{-1/3}$ and $\epsilon = R_e^{-2/3}$. The buoyancy term is of the order $R_e^{-(4/3-n)}$. So the effect of stratification on the jet flow depends upon the value of n. This effect is large or small according as $n \gtrless 4/3$. As compared to Kelley and Redkopp and Redkopp's cases, the critical stratification for the jet problem is larger than for the flow over a flat plate.

<u>CASE</u> (i) $n < 4/3$

The inner expansions

$$\Psi = \Psi^{(1)} + \sigma(R_e)\,\Psi^{(2)} + \cdots$$

$$T^* = T^{(1)} + \gamma(R_e)\,T^{(2)} + \cdots$$

when substituted in equations (6) lead to

$$\left(\Psi_z^{(1)} \frac{\partial}{\partial x} - \Psi_x^{(1)} \frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\right) \frac{\partial^2}{\partial z^2} \Psi^{(1)} = 0$$

$$\left(\Psi_z^{(1)} \frac{\partial}{\partial x} - \Psi_x^{(1)} \frac{\partial}{\partial z} - \frac{1}{P_r} \frac{\partial^2}{\partial z^2}\right) T^{(1)} = 0 \qquad (8)$$

In the above considerations P_r is considered to be of order unity. Equations (8) are the familiar equations for a classical jet. Similarity solutions of these equations have been obtained by Schlicting and Yih. We notice that for

$n < 4/3$, the effect of stratification on the jet flow is of higher order. In order to obtain the higher-approximations, we need the first order outer flow solution.

OUTER SOLUTION

Following the usual procedure the outer expansion for ψ & T^* can be written as:

$$\psi(x,y,Re) = Re^{-1/3}\,\psi_1(x,y) + \cdots$$

$$T^*(x,y,Re) = T_1(x,y)$$

The zeroth order outer solution is taken to be zero. Substituting these expressions in equations (3), we get

$$T_1(x,y) = \beta(\psi_1 - y) + C_1 \qquad (i) \qquad (9)$$

and

$$\left(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}\right)\nabla^2\psi_1 + \alpha\, Re^{n+\frac{2}{3}}\,\psi_{1x} = 0 \qquad (ii)$$

once again, the Solution of (9-ii) depends upon the value of n and is given as:

$$\nabla^2\psi_1 = 0 \qquad \text{if} \quad n < -2/3 \qquad (i) \qquad (10a)$$

$$\psi_{1x} = 0 \qquad \text{if} \quad n > -2/3 \qquad (ii)$$

for $n = -2/3$, we get

$$\left(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}\right)\nabla^2\psi_1 + \alpha\,\psi_{1x} = 0 \qquad (10b)$$

It may be noticed that for $4/3 > n > -2/3$, baroclinic generation of vorticity in the outer inviscid flow becomes important to this order of analysis. For $n > -2/3$ the solution of equation (10-ii) does not match with the first order boundary-layer solution of Schlicting and as such the second order correct solution requires the introduction of an intermediate layer. It will be shown that the governing equation for this layer is similar to equation (10b) and for $n = -2/3$ no such layer is required. In order to satisfy the boundary conition on temperature, the solution of euqations (10) should be of the form

$$\psi_1 = y + \psi_\alpha$$

which leads to a finite velocity at infinity. In other words the displacement of stream lines due to first order boundary layer induces a small velocity $\sim R_e^{-1/3}$ in the outer flow. This anomalous result is due to the linear model of stratification used in the present investigation which, in fact, at large distances breaks down. This break down of the model is characteristic of zero or non-uniform free stream velocity. This difficulty could be avoided by satisfying the outer boundary condition on temperature at some finite distance say $y = (\alpha_o \beta_o \tau_o)^{-1}$. This choice is compatible with the linear model for stratification. The arbitrary constant C_1 in (9-i) could be choosen to be $9(\alpha_o \beta_o \tau_o)^{-1}$. Now, for $n < -2/3$, the outer solution is same as obtained by Rubin and Falco and the effect of stratification on jet flow is shifted to higher order approximations.

INTERMEDIATE-LAYER

For $4/3 > n > -2/3$ and intermediate layer is required. The equation for this layer is obtained by rescaling the variables as:

$$\hat{x}, \hat{y} = \frac{\sigma(R_e)}{\epsilon(R_e)} (x, y) \qquad \sigma(R_e) \ll 1$$

$$\psi = \epsilon/\sigma \; \hat{\psi}(\hat{x}, \hat{y}) \tag{11}$$

$$T^* = \epsilon/\sigma \; \theta(\hat{x}, \hat{y})$$

The purpose of this rescaling is to obtain a balance between convective and buoyancy terms. The following expansions

$$\hat{\psi} = \delta^*(Re)\,\hat{\psi}_1 + \cdots$$
$$\theta = \theta_1 + \cdots$$

when substituted in the governing equations, lead to

$$\left.\begin{aligned} &\theta_1 = \beta(\hat{\psi}_1 - \hat{y}) \\ &\left(\hat{\psi}_{1\hat{y}} \frac{\partial}{\partial \hat{x}} - \hat{\psi}_{1\hat{x}} \frac{\partial}{\partial \hat{y}}\right)\left(\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2}\right)\hat{\psi}_1 + \alpha \hat{\psi}_{1\hat{x}} = 0 \end{aligned}\right] \quad (12)$$

and

$$(\epsilon/\sigma)^2\, Re^n/\delta^{*2} = 1 \quad (13)$$

Equation (12-ii) is same as (10-b). The boundary conditions for (12) can be obtained by matching with the inner boundary-layer and outer flow. This leads to

$$\hat{\psi}(\hat{x},0) = 2\alpha^* \hat{x}^{1/3} \quad \text{and} \quad (14)$$

$$\left.\begin{aligned} \epsilon/Re &= Re^{-1/5\,(1+\frac{3}{2}n)} \\ \delta^* &= Re^{-1/3}\, Re^{-\frac{1}{5}(1+\frac{3}{2}n)} \end{aligned}\right] \quad (15)$$

For n = -2/3, $\epsilon = \sigma$ and the rescaling of the variables is not required and equations (10-b) and (12) appropriately match with the boundary-layer solution. (12-ii) can be integrated along a stream line and we get

$$\nabla^2 \hat{\psi}_1 + \alpha \hat{\psi}_1 = \alpha \hat{y} \quad (16)$$

The last equation can be solved as

$$\hat{\psi}_1 = \hat{y} + \hat{\psi}_a \quad \text{where} \quad (17)$$

$$\left.\begin{aligned} &\nabla^2 \hat{\psi}_a = \alpha \hat{\psi}_a \\ &\hat{\psi}_a(\hat{x},0) = 2\alpha^* \hat{x}^{1/3} \end{aligned}\right] \quad (18)$$

The fundamental solution for (18) has been obtained by Graham. The solution for $\hat{\psi}_\alpha$ is given as

$$\hat{\psi}_\alpha = 2\alpha^* \int_{-\infty}^{\infty} \xi^{1/3} \left[\frac{\partial G}{\partial \eta} - \frac{\partial G^*}{\partial \eta} \right]_{\eta=0} d\xi \qquad (19)$$

where

$$G(x,y,\xi,\eta) = N_1(\sqrt{\alpha}\, r) \sin\theta + \frac{1}{\pi} \sum_n \frac{8n}{4n^2-1} J_{2n}(\alpha^{\frac{1}{2}} r) \sin 2n\theta$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2$$

The expression for G^* is the same as for G but with r replaced by r^* where

$$r^{*2} = (x-\xi)^2 + (y+\eta)^2$$

Without actually calculating the integral in (19) one can easily see that the solution (17) does match to the outer flow. With the expansion for the intermediate-layer obtained the gauge function for the second order boundary-layer equations can now be obtained. These equations come out to be the same as obtained by Rubin and Falco for $n < 34/33$. For $4/3 > n$ 34/33 matching requires the introduction of another sub layer. Because of the outer boundary condition for the second-order boundary-layer equations (from matching with intermediate layer) the similarity nature of the solution of Rubin and Falco is not retained and a numerical solution to the problem is required.

BUOYANCY DOMINATED JET

For $n = 4/3$, the buoyancy term in the first order boundary-layer equations is of order unity. This also represents a cross-over point where the jet is no more inertia dominated. The pressure term in (7) also becomes of the same order as velocity. A similarity solution for this case seems doubtful and a full numerical solution should be sought. For $n > 4/3$ the jet is buoyancy dominated. The structure of the jet is determined through a balance between buoyancy and viscous terms. This requires

$$\frac{\delta}{\epsilon^4 R_e^{n+1}} \sim O(1) \qquad (20)$$

The constancy of momentum flux is determined by the pressure term in equation (7) and is no more the one obtained previously. As shown by Pao and Martin and Long this condition requires

$$\epsilon = \delta^4 \qquad (21)$$

From (20) and (21), we get

$$\epsilon \sim R_e^{-4/15\,(n+1)}$$

The convective terms are of $\sim R_e^{-1/3\,(n-2)}$ and so for $n > 2$ are negligible. The region $4/3 < n < 2$ is such where convection, buoyancy and viscosity are important and the equations similar to the case for $n = 4/3$ should be solved. For very large Prandtl numbers, these equations lead

$$\frac{\partial^4}{\partial z^4}\psi^{(1)} = -\beta\,\psi^{1}_{x} \qquad (22)$$

The boundary conditons are

$$\psi^{(1)}(x,0) = \psi^{(1)}_{zz}(x,0) = 0 \qquad (23)$$
$$\psi^{(1)}_{z}(x,z) \to 0 \quad \text{as } z \to \infty$$

Equation (22) has already appeared in the works of Martin and Long and Pao. The solution can be obtained in similarity form numerically.

REMARKS

From mathematical standpoint the introduction of a fictitious wall to satisfy the boundary condition is questionable. The solution along the lines of Murray's analysis of shear flow should be worked out, and the role of linear stratification in an unbounded fluid clarified.

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